and curves, are followed by Chapter 4, where the whole (subject) is again explained "from Adam" by defining the concept of a function, explaining the meaning of the coordinate system and the methods of function representation in the form of curves, and by giving examples of the simplest functions and their respective curves.

The unprofessional style of presenting mathematics and laws of mechanics without any qualification as regards the limits of validity of various statements is a feature of the entire content of the book. This is not the way to develop inquisitiveness of the reader, since he is deprived of the opportunity to obtain a real understanding of the essence of the subject. Worst of all, such style may lead the "beginner" or the nonspecialist to the illusion of understanding.

A. A. Dorodnitsyn, L. S. Pontriagin and L. I. Sedov

AUTHOR'S REPLY

ON THE TEACHING OF HIGHER MATHEMATICS AND MY BOOK "HIGHER MATHEMATICS FOR BEGINNERS AND ITS APPLICATION IN PHYSICS"

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It is perhaps for the first time that the subject of teaching mathematics appears on the pages of PMM, and I am glad to take this opportunity for presenting my views on it.

In defining the main purpose of teaching mathematics primary consideration must be given to people who will apply it in practice and not to professional teachers, but the latter must have a decisive influence on the elaboration of teaching methods.

I consider the historical approach as the most important principle which is to be taken into consideration in the broadest formulation of teaching. The student should be led through the stages (of science development) which were passed by humanity (memorization of dates and names is not necessary). In many instances one has to have the courage to renounce clearly at the beginning of a course of lectures the latest, more e fashionable, and more rigorous treatments recently developed.

The second general principle is the realization that understanding and creative assimilation of new concepts occur intuitively and are enhanced by practical applications. The introduction of new concepts by rigorous, formally and logically faultless definitions and proofs is pedagogically unsound. The faultlessness will not be appreciated by a person who only begins to get familiarized with a new branch of science. The importance of strictness in the development of science itself and of reverting, after the first intuitive concenter (stage), to fundamentals from strictly defined positions is not denied.

I consider that theaching of higher mathematics must begin in practice with the introduction of notions of the derivative and of the integral, omitting the theory of limits.

Obviously such approach is not rigorous, since the concepts of the derivative and of the integral are based on some specific passing to limit. It is not without fault, since a passing to limit is not always possible and does not always lead to a definite quantity. Although conscious of all this, I nevertheless consider that at the initial teaching stage attention must be fixed on positive content of the notions of the derivative and of the integral without stressing exceptional cases (discontinuous functions, etc.) and disregarding the subject of limits.

The concept of instantaneous velocity as the derivative of a coordinate with respect to time, that of the tangent to a curve and of the tangent of the angle of inclination of a tangent constitute the material for assimilating the concept of the differential. Let us explain this on the example of $y(x) = x^2$. Since $\Delta y/\Delta x = [y(x + \Delta x) - y(x)]/\Delta x = 2x + \Delta x$, hence passing to the limit $\Delta x \to 0$, we obviously obtain $y' = dy/dx = 2x$. The student has been given material for the assimilation, development, and (most important) for the application of derivatives. The same applies to the concept of the integral. Relation between the derivative and the integral is clarified. The range of considered functions is extended from polynomials of integral powers to those of other powers and to exponential functions; the number e is defined by the condition $de^x/dx = e^x$, and trigonometric functions are considered.

Let us recall the history of science which shows that an approximately similar set of data had opened to the eighteenth and nineteenth century scientists a huge field for the application of mathematics. Let us not derive our youth of the chance of retracing that path.

In his authobiography Einstein wrote: "At the age of 12–16 I became acquainted with elements of mathematics, including the fundamentals of the differential and integral calculus. I was fortunate to come across books in which not too much emphasis was put on strict logic, while fundamental ideas were clearly expounded. The whole process was truly absorbing; there were instants of elation no less impressive than the "miracle" of elementary geometry, the basic idea of analytic geometry, infinite series, and the concept of the differential and of the integral".

Hence it is necessary to consider in the first instance the basic question of the expediency of providing a course and a program as far as possible free of formalism. One would be inclined to bypass the theory of limits at the early stages of familiarization with higher mathematics.

I consider it incontrovertible that such plan of action has the right to exist side by side with conventional programs. There remain other important questions, although secondary in comparison with those considered above, such as:

1) what examples are to be used in theaching the reader to apply concepts of higher mathematics in practical problems;

2) to what extent are numerical methods to be used;

3) when and how is the notion of limits to be defined more rigorously (I favor the positive approach, namely the introduction of the Dirac delta function and its related concepts).

Not only the general scheme and the pedagogical principles, but also the method of embodiment of the latter are legitimate subjects for examination and criticism.

My ideas on the solution of these questions are embodied in my book "Higher Mathematics for Beginners and Its Application in Physics".

Its first two editions appeared in 1958. On the recommendation of the Scientific Council of the Institute of Applied Mathematics the third edition was approved by the Ministry of Education of the USSR as an optional textbook.

The fourth and fifth editions were published in 1968 and 1970. Some rearrangements, alterations and additions were included in these. A Chapter on the delta function and a brief very general introduction to certain fields of mathematics closely related to physical theories but lying outside the scope of an elementary course were added.

The review by Academicians A. A. Dorodnitsyn, L.S. Pontriagin and L. I. Sedov, which appears above, was originally written in connection with the consideration by the Publishers of the advisability of publishing a sixth edition of the book.

Unfortunately the review does not deal with the really fundamental questions outlined above. The actual need for a textbook of a new kind is evidently so great that the Reviwers do not deny it. This, however, does not settle the problem of the book. The latter which is the embodiment of certain specific ideas can and must be undoubtedly discussed and criticized.

It is perfectly legitimate to point out specific errors and to criticize shortcomings in the execution of the conceived plan.

It is not for me to decide whether the book is written "carelessly", is "verbose" and "confusing" (review) or is "lively" and "absorbing" (verdict of the Scientific Council of the Instutute of Applied Mathematics).

Some of the remarks appearing in the review should be taken into account, if a further edition of the book would be contemplated. Thus, for instance, the position in the fifth edition of the Chapter on curves and functions in the middle of the book is unfortunate. The Chapter should be returned to the position it occupied in the fourth edition, i. e. at the beginning of the book.

Should a certain increase of the book size be contemplated, it would be desirable to introduce the notion of the inertial system of coordinates.

I may further suggest that it would not be out of place to point out the similarity between the situation in the gravitational field and in an accelerating system, thus introducing the reader to the origins of the general theory of relativity.

It would be advisable to treat the subject of drag of bodies moving in a liquid or gas in greater detail (this subject which attracted the particular attention of the Reviewers is dealt with below). I would also suggest that in the Section on light absorption mention is made of light amplification in lasers.

The book undoubtedly can be improved, and some of the Reviewer's remarks, unfortunately few, can help in this. However I wish to avoid giving the impression that the review and my answer to it conform to the normal process of book review.

I consider the general tone of the review totally inadmissible and am astonished by the attitude of the Editors who let it appear in print. Let me point out the review lack of objectivity on a few examples.

In relation to p. 17 the Reviewers write: "This statement is clearly not always true". However later in the book it is clearly stated that the limit exists only for for points at which the curve is smooth. The Reviewers, while omitting any mention of this, write that a suitable qualification would have "immediately provided a clearer understanding", a statement whose essence I do not accept. It seems to me that an "immediate" inclusion of limits is inadvisible, exercises must come first. It is clear that the question is not of negligence but of different pedagogical concepts.

As regards pp. 525 and 526 the Reviewers overlook the fact that the offending passages do not deal with specific theories but are intended as a guide to the reader.

The theory of curvilinear coordinates in a Euclidean space is an introduction to the non-Euclidean geometry which is at the basis of the remarkable achievement of contemporary science: the general theory of relativity. I am convinced that the reader will understand this idea, and in writing this book I did not set any higher aim.

Finally, I shall deal with the problem of drag of a body moving in a liquid or gas, the problem which attracted the most crude attack of the Reviewers.

The derivation of the formula for drag, which appears on p. 350 goes back to Newton: a body of cross section area S moving at velocity V displaces in a unit of time the volume SV of the fluid; if the displaced fluid attains velocity V, its kinetic energy becomes equal to $SV\rho.\ V^2/2$ and the corresponding force is $F = S\rho V^2/2$, a result that differs from the true by the dimensionless coefficient K. This was known to Newton, is known to me, and is known to the Reviewers. It is precisely this that I intended to communicate to readers.

Now follows the Reviewers' strange statement: "If the correct procedure is followed in the case of well streamlined bodies, in the absence of viscosity, the d'Alembert paradox of zero drag is obtained".

Yes, Newton did not know that, but the Reviewers and I know that in the case of zero viscosity a formal solution with zero drag is obtained. However we know more, namely, that in the case of stationary flow past bodies we have instead of that solution a turbulent one to which Newton's formula (with nearly constant $\varkappa \neq 1$) is applicable. Why then, call "correct" something that does not occur in nature? Is it only for hurting me deeper? Do I have to explain the d'Alembert paradox and then refute it in a book where the law of drag is primarily needed for deriving the law of motion of a body?!

Let me deal with the second subject, that of the dependence of force (drag) on viscosity, which is equivalent to the effect of the Reynolds number on the drag coefficient. I feel somewhat embarassed to have to quote on the pages of PMM a few figures (for a sphere), namely, that for $\mathrm{Re} = 100$, $\varkappa = 1.2$, for $R = 2 \cdot 10^5$, $\varkappa = 0.4$, and beyond the critical region of Reynolds number (at least) up to $\mathrm{Re} = 10^8$, $\varkappa = 0.12$; thus a change of Re by six orders of magnitude alters $\varkappa$ by one order. A rough interpolation yields $\varkappa \sim \mathrm{Re}^{-1/6}$ and for the force (drag) $F \sim \nu^{1/6}$. By this interpolation I open myself to further criticism, since this relation is in fact nonexponential. However, considering the purpose of my book, am I not right in saying that the esponent 1/6 defines a weak dependence?! The analysis of the motion of bodies on the assumption of $\varkappa = \mathrm{const}$ and $F \sim v^2$ is generally accepted in textbooks on mechanics; my definition of drag as being "virtually independent" fully describes the situation.

I would draw the attention to an error in the review which may be due to negligence, or be deliberately illogical. The Reviewers rightly note that in the case of well streamlined bodies up to 85% of the drag is due to viscosity, i.e. is determined by integrating over the whole surface of the body the product of viscosity ν by the velocity gradient (the force is integrated as a vector). Thus $F = a + b\nu$, where the first term represents the contribution (to drag) of pressure equal to $0.15F$ and the second which represents the 85% is $b\nu = 0.85F$. Is this equivalent to the assertion that the force (drag) F strongly depends on viscosity? If b were constant, we would have $d\ln F / d\ln \nu = 0.85$ and the effective exponent $F \sim \nu^{0.85}$.

In reality the pattern of a turbulent flow alters with increasing ν, while the velocity gradient decreases so that the term $b\nu$ as a whole weakly depends on ν, if the change of the flow pattern is taken into account.

Thus the true statement about the considerable effect of viscosity does not contradict the statement in my book that drag is weakly affected by viscosity.

The motive for writing a review in which "negligence" is the mildest and "provocation" the most colorful expression, and the main criticism of a book for beginners relates to lack of subtlety and of details is altogether not clear to me.

Ia. B. Zel'dovich